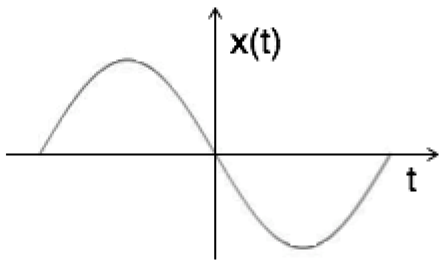
SIGNALS SAMPLING THEOREM

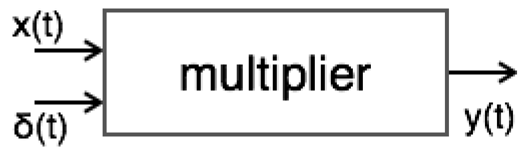
Statement: A continuous time signal can be represented in its samples and can be recovered back when sampling frequency f_s is greater than or equal to the twice the highest frequency component of message signal. i. e.

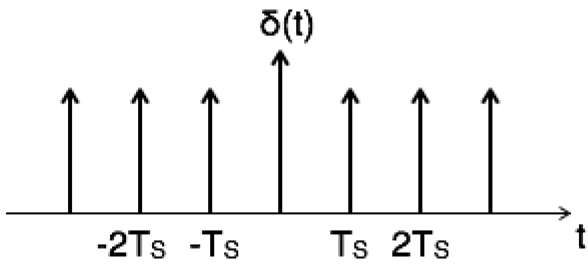
$$f_s \leq 2f_m.$$

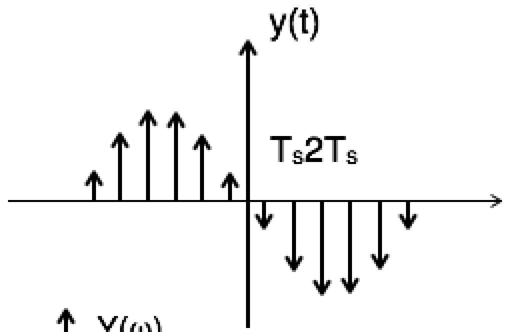
Proof: Consider a continuous time signal xt. The spectrum of xt is a band limited to f_m Hz i.e. the spectrum of xt is zero for $|\omega|>\omega_m$.

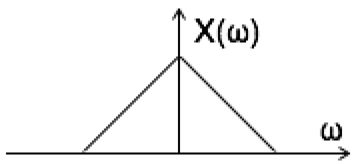
Sampling of input signal xt can be obtained by multiplying xt with an impulse train δt of period T_s. The output of multiplier is a discrete signal called sampled signal which is represented with yt in the following diagrams:

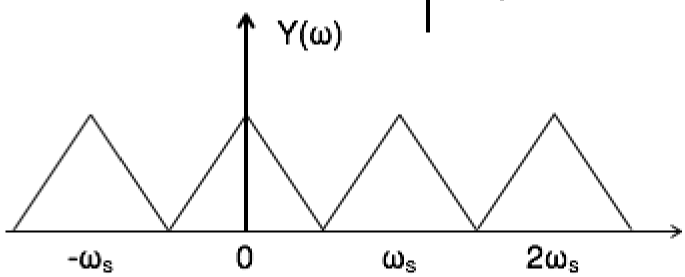












Here, you can observe that the sampled signal takes the period of impulse. The process of sampling can be explained by the following mathematical expression:

$$\text{Sampled signal}\, y(t) = x(t).\, \delta(t) \;\ldots\ldots(1)$$

The trigonometric Fourier series representation of δt is given by

$$\delta(t) = a_0 + \Sigma_{n=1}^{\infty}(a_n \cos n\omega_s t + b_n \sin n\omega_s t) \;\ldots\ldots(2)$$

Where $a_0 = \frac{1}{T_s}\int_{\frac{-T}{2}}^{\frac{T}{2}} \delta(t)dt = \frac{1}{T_s}\delta(0) = \frac{1}{T_s}$

$$a_n = \frac{2}{T_s}\int_{\frac{-T}{2}}^{\frac{T}{2}} \delta(t)\cos n\omega_s\, dt = \frac{2}{T_2}\delta(0)\cos n\omega_s 0 = \frac{2}{T}$$

$$b_n = \frac{2}{T_s}\int_{\frac{-T}{2}}^{\frac{T}{2}} \delta(t)\sin n\omega_s t\,dt = \frac{2}{T_s}\delta(0)\sin n\omega_s 0 = 0$$

Substitute above values in equation 2.

$$\therefore\ \delta(t) = \frac{1}{T_s} + \Sigma_{n=1}^{\infty}(\frac{2}{T_s}\cos n\omega_s t + 0)$$

Substitute δt in equation 1.

$$\rightarrow y(t) = x(t).\,\delta(t)$$

$$= x(t)[\frac{1}{T_s} + \Sigma_{n=1}^{\infty}(\frac{2}{T_s}\cos n\omega_s t)]$$

$$= \frac{1}{T_s}[x(t) + 2\Sigma_{n=1}^{\infty}(\cos n\omega_s t)x(t)]$$

$$y(t) = \frac{1}{T_s}[x(t) + 2\cos\omega_s t.\,x(t) + 2\cos 2\omega_s t.\,x(t) + 2\cos 3\omega_s t.\,x(t)\ldots\ldots]$$

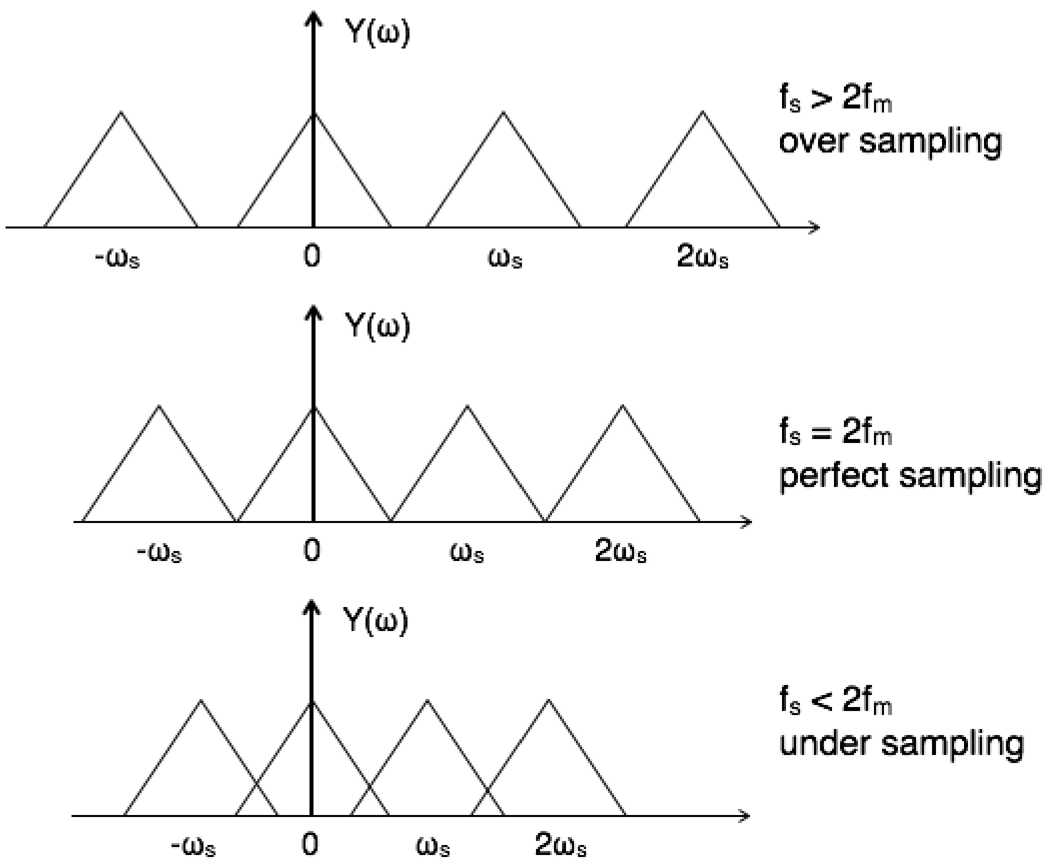
Take Fourier transform on both sides.

$$Y(\omega) = \frac{1}{T_s}[X(\omega) + X(\omega-\omega_s) + X(\omega+\omega_s) + X(\omega-2\omega_s) + X(\omega+2\omega_s)+\ldots]$$

$$\therefore\ Y(\omega) = \frac{1}{T_s}\Sigma_{n=-\infty}^{\infty}X(\omega - n\omega_s) \qquad where\ n = 0, \pm1, \pm2, \ldots$$

To reconstruct xt, you must recover input signal spectrum Xω from sampled signal spectrum Yω, which is possible when there is no overlapping between the cycles of Yω.

Possibility of sampled frequency spectrum with different conditions is given by the following diagrams:



Aliasing Effect

The overlapped region in case of under sampling represents aliasing effect, which can be removed by

- considering $f_s > 2f_m$
- By using anti aliasing filters.